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Published in:
Game theory and economic applications

Publication date:
1992

GAMES OF COORDINATION

by

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ABSTRACT. Two person games are studied which consist of several smaller games that can only be reached if both players coordinate. It is assumed that both players gain from coordinating and are indifferent otherwise. Correspondences are given between equilibria for such a game and equilibria for the smaller games of which it consists. The same is done with respect to equilibria that are stable against small mistakes the players could make, i.e. that are perfect or contained in a stable set.

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1. INTRODUCTION

In many situations of a conflict between two individuals there is only question of a real conflict, i.e. one that really affects the individuals, if the two individuals *coordinate*, which means, for example, meet at the same place, or at the same time.
Typical examples of conflicts where coordination is important form the so-called 'Blotto games' (Tuckley (1949), Blacket (1954)). These games are about two opposing armies which can meet at the same battlefield or not. With respect to these games Luce and Raiffa (1957) make the following remark (p455): "An interesting mathematical problem connected with these games is the determination of their solution from the payoff functions of the individual battlefields".
In this paper we deal with this problem in the case where both players gain from coordinating and are indifferent when they mismatch. We look at games where both players can choose from finitely many alternatives, i.e. bimatrix games.
One evening a man and a woman, living together, have to decide whether to have dinner at home (h) or at the restaurant around the corner (r). Each of them has to choose between the two alternatives independent of the other person. However, both like to have dinner together, so that they only gain when they choose the same alternative. So far our example is just a variant of what is known as 'battle of the sexes' (cf. Luce and Raiffa). But here we make the situation more complicated. We assume the restaurant serves two menus ($m1$ and $m2$) and that if both the man and the woman decide to go to the restaurant, then they like more to have different menus more than to have the same menu (they like to exchange some of the food). Further, since they have a traditional way of life, which means that the woman does the cooking, she has a larger interest of going to the restaurant than the man. Thus the following game reflects their conflict

$$\begin{array}{ccc} & & \text{woman} \\ & & \begin{array}{ccc} h & m1 & m2 \end{array} \\ \text{man} & \begin{array}{c} h \\ m1 \\ m2 \end{array} & \begin{pmatrix} (2,1) & (0,0) & (0,0) \\ (0,0) & (1,2) & (2,3) \\ (0,0) & (2,3) & (1,2) \end{pmatrix} \end{array}$$

Some equilibria for this game are (h,h), $(rm1, rm2)$ and $(rm2, rm1)$, since, as the equilibrium concept requires, unilateral deviation from these choices is not profitable. Clearly the whole game has only two subgames that are important. They are the 'restaurant game', which is the game that remains if both decide to go to the restaurant and the 'home game', which is not really a game, since it leaves no choice to the players. In section 3 we show that by solving these conflicts first we can easily solve the whole game. In fact we show that we can also walk in the opposite direction. In section 4 we show that the results in section 3 also hold if we consider more restricted solutions than equilibria, namely if also we require stability against mistakes. We consider perfect equilibria (Selten (1975)) and stable sets (Kohlberg and Mertens (1986)). In section 5 we make some final remarks. Section 2 is preliminary.

NOTATIONS By $\mathbb{N}$ we denote the set $\{1, 2, \ldots\}$. For $t \in \mathbb{N}$ we denote by $\mathbb{R}^t$ the set of t-tuples of real numbers. We let $e_1, e_2, \ldots, e_t$ be the standard basis vectors in $\mathbb{R}^t$. For $x, y \in \mathbb{R}^t$ we write $x \geq y$ if $x_i \geq y_i$ for all $i \in \{1, 2, \ldots, t\}$.

2. BIMATRIX GAMES

Let $M := \{1, \ldots, m\}$ and $N := \{1, \ldots, n\}$. Let $A : [a_{ij}]_{i \in M, j \in N}$ and $B := [b_{ij}]_{i \in M, j \in N}$ be two real $m \times n$ matrices. The $m \times n$ *bimatrix game* (A, B) is defined as the two-person game where player 1 and player 2 independently choose *strategies* $p \in \Delta_m$ and $q \in \Delta_n$ respectively and accordingly obtain payoffs $pAq := \sum_{i=1}^m \sum_{j=1}^n p_i a_{ij} q_j$ and $pBq = \sum_{i=1}^m \sum_{j=1}^n p_i b_{ij} q_j$. A pair of strategies $(p, q) \in \Delta_m \times \Delta_n$ is called an *equilibrium* for (A, B) if $xAq \leq pAq$ and $pBy \leq pBq$ for all pairs of strategies $(x, y) \in \Delta_m \times \Delta_n$. The set of equilibria for (A, B) is denoted by $E(A, B)$. For a strategy $p \in \Delta_m$ we denote by $C(p) := \{i \in M \mid p_i > 0\}$ the *carrier* of p and by $PB(B, p) := \{j \in N \mid pBej = \max_{l \in N} pBe_l\}$ the *pure best replies* against p. For $q \in \Delta_n$, $C(q)$ and $PB(A, q)$ are defined analoguously. It is a well-known fact that if $(p, q) \in \Delta_m \times \Delta_n$, then $(p, q) \in E(A, B)$ if and only if $C(p) \subset PB(A, q)$ and $C(q) \subset PB(B, p)$. Nash (1951) showed that $E(A, B)$ is nonempty for all games (A, B).

Selten (1975) introduced the idea of stability against small mistakes the players could make in choosing their strategies and thus defined perfect equilibria. His idea can be modelled with help of so-called perturbed games.

Let (A, B) be an $m \times n$ bimatrix game. For a *mistake vector* $(\varepsilon, \delta) \in \mathbb{R}^m \times \mathbb{R}^n$, i.e.

$$\varepsilon > 0, \sum_{i=1}^m \varepsilon_i < 1 \quad \text{and} \quad \delta > 0, \sum_{j=1}^n \delta_j < 1,$$

the (ε, δ)*-perturbed game* $(A, B, \varepsilon, \delta)$ corresponding to (A, B) is defined to be the game which only differs from (A, B) in the sense that the strategy spaces are restricted to

$$\Delta_m(\varepsilon) := \{p \in \Delta_m \mid p \geq \varepsilon\} \quad \text{and} \quad \Delta_n(\delta) := \{q \in \Delta_n \mid q \geq \delta\}.$$

Equilibria for the perturbed game $(A, B, \varepsilon, \delta)$ can be defined in an obvious way.

Now an equilibrium (p, q) for (A, B) is called *perfect* if there exist a sequence $\{(\varepsilon^k, \delta^k)\}_{k \in \mathbb{N}}$ of mistake vectors in $\mathbb{R}^m \times \mathbb{R}^n$ converging to zero and a sequence $\{(p^k, q^k)_{k \in \mathbb{N}}$ converging to (p, q) such that (p^k, q^k) is an equilibrium for $(A, B, \varepsilon, \delta)$ for every $k \in \mathbb{N}$.

Selten showed that for every bimatrix game there is a perfect equilibrium.

An equilibrium which is stable against all small mistakes the players could make is called *strictly perfect* (Okada (1981)) and defined in an obvious way. However, van Damme (1987) gave an example of a bimatrix game without a strictly perfect equilibrium. In Kohlberg and Mertens (1986) the set-valued equivalent of a strictly perfect equilibrium is defined. Below we give a definition equivalent to the one of Kohlberg and Mertens. First we define for an $m \times n$ bimatrix game (A, B) and a mistake vector $(\varepsilon, \delta) \in \mathbb{R}^m \times \mathbb{R}^n$ the perturbed matrices $A(\delta)$ and $B(\varepsilon)$ by

$$e_i A(\delta) := \left(1 - \sum_{j=1}^n \delta_j\right) e_i A + (e_i A \delta) 1_n \quad \text{for} \quad i \in M$$

and

$$B(\varepsilon)e_j := \left(1 - \sum_{i=1}^{m} \varepsilon_i\right) Be_j + (\varepsilon Be_j)1_n \quad \text{for} \quad j \in N,$$

where, for $t \in \mathbb{N}$, $1_t \in \mathbb{R}^t$ is the vector with all coordinates equal to 1.

For an $m \times n$ bimatrix game (A, B) we call a closed set $T \subset E(A, B)$ *strictly perfect* if for every sequence $\{(\varepsilon^k, \delta^k)\}_{k\in\mathbb{N}}$ of mistake vectors in $\mathbb{R}^m \times \mathbb{R}^n$ converging to zero there is a sequence $\{(p^k, q^k)\}_{k\in\mathbb{N}}$, with a limit point in T such that $(p^k, q^k) \in E(A(\delta^k), B(\varepsilon^k))$ for every $k \in \mathbb{N}$.

A strictly perfect set that does not properly contain another strictly perfect set is called a *stable set*. In Jansen et al. (1990) it is shown that every stable set for a bimatrix game is finite and that there exists a stable set for every bimatrix game. The latter result was already obtained by Kohlberg and Mertens for n-person games.

It is easy to show (cf. Jansen et al (1990)) that an equilibrium (p, q) for an $m \times n$ bimatrix game (A, B) is perfect if and only if there is a sequence $\{(\varepsilon^k, \delta^k)\}_{k\in\mathbb{N}}$ of mistake vectors in $\mathbb{R}^m \times \mathbb{R}^n$ converging to (p, q) such that $(p^k, q^k) \in E(A(\delta^k), B(\varepsilon^k))$ for every $k \in \mathbb{N}$. This shows that every equilibrium contained in a stable set is perfect.

3. EQUILIBRIA FOR A GAME OF COORDINATION

In the following we let the $m \times n$ bimatrix game (A, B) be a game of coordination. This implies that (A, B) is the composition of a number of, say t, conflicts which are also bimatrix games. The latter games we denote by (A^1, B^1), $(A^2, B^2), \cdots, (A^t, B^t)$, where for $s \in \{1, \cdots, t\}$ we suppose that (A^s, B^s) is an $m_s \times n_s$ bimatrix game. When the players actually play one of these games they have coordinated well and we assume that they gain from that. This implies that for every $s \in \{1, \cdots, t\}$ all entries of A^s and B^s are positive. When the players do not coordinate well they obtain a payoff zero, so that all entries in A en B are nonnegative.

In the example of section 1 we dealt with a 3×3 game of coordination (A, B), which is the composition of a 1×1 game (A^1, B^1) called the 'home game' and a 2×2 game (A^2, B^2) called the 'restaurant game'. Clearly the home game is not really a game (in this game both players have only one strategy).

The restaurant game $(A^2, B^2) = \begin{bmatrix} (1,2) & (2,3) \\ (2,3) & (1,2) \end{bmatrix}$ has three equilibria: $(m1, m2)$, $(m2, m1)$ and $(\frac{1}{2}m1 + \frac{1}{2}m2, \frac{1}{2}m1 + \frac{1}{2}m2)$.

Assume that *if* the players meet in the restaurant, then $(m1, m2)$ is played and $(2, 3)$ is the outcome. Then in the whole game (A, B) we can replace the restaurant game by $(2, 3)$ so that it becomes $\begin{bmatrix} (2,1) & (0,0) \\ (0,0) & (2,3) \end{bmatrix}$.

This game has three equilibria: (h, h), (r, r) and $(3/4h + 1/4r, 1/2h + 1/2r)$. Note that for the man (player 1) r actually means $rm1$ and for the woman $rm2$. Then it is easy to check that the equilibria above are also equilibria of the original game (A, B). We show that with a generalization of this procedure all equilibria can be found.

Clearly we can repesent the $m \times n$ game (A, B) by

$$A = \begin{bmatrix} A^1 & & & \\ & A^2 & & \varnothing \\ & & \ddots & \\ & \varnothing & & \\ & & & A^t \end{bmatrix} \text{ and } B = \begin{bmatrix} B^1 & & & \\ & B^2 & & \varnothing \\ & & \ddots & \\ & \varnothing & & \\ & & & B^t \end{bmatrix}.$$

Note that $m = \sum_{s=1}^{t} m_s$ and $n = \sum_{s=1}^{t} n_s$. From now on we refer to (A, B) as the *major game* and to (A^1, B^1), $(A^2, B^2), \cdots, (A^t, B^t)$ as subgames. Before we prove the first results we give some more definitions:
For $x \in \mathbb{R}^m$ and $s \in \{1, \cdots, t\}$ we denote

$$x(s) := (x_{m_1+\cdots+m_{s-1}+1}, \cdots, x_{m_1+\cdots+m_{s-1}+m_s}) \in \mathbb{R}^{m_s}$$

and

$$M_s := \{m_1 + \cdots + m_{s-1} + 1, \cdots, m_1 + \cdots + m_{s-1} + m_s\}.$$

Similarly, for $y \in \mathbb{R}^n$ and $s \in \{1, \cdots, t\}$, $q(s)$ and N_s are defined. Further for $i \in M_s$ for some $s \in \{1, \cdots, t\}$ we let $i(s) := i - \sum_{r=1}^{s-1} m_s$ and similarly for $j \in N_s$, $j(s) := j - \sum_{r=1}^{s-1} n_s$.

Now let (p, q) be an equilibrium for the major game. Take $s \in \{1, \cdots, t\}$ and suppose $p(s) \neq 0$. Then there must be an $i \in M_s$ such that $p_i > 0$. Since $C(p) \subset PB(A, q)$, this implies that $e_i A q = \max_{k \in M} e_k A q$. Clearly $q(r) \neq 0$ for at least one $r \in \{1, \cdots, t\}$. Since all entries of A^r are positive for every r, we obtain that $\max_{k \in M} e_k A q = \max_{r \in \{1,\cdots,t\}} \max_{k \in M_r} e_{k(r)} A^r q(r) > 0$. Hence $e_i A q > 0$. Since $e_i A q = e_{i(s)} A^s q(s)$ and since A^s has only positive entries, this implies $q(s) \neq 0$.
Similarly, for $s \in \{1, \cdots, t\}$, $q(s) \neq 0$ implies $p(s) \neq 0$. So

LEMMA 3.1. Let (p, q) be an equilibrium for the major game. Then, for every $s \in \{1, \cdots, t\}$, $p(s) \neq 0$ if and only if $q(s) \neq 0$.

For $x \in \mathbb{R}^m$ and $s \in \{1, \cdots, t\}$ we define the 'balanced projection of x on $\mathbb{R}^{m_s}$', $\Pi^s(x)$, by

$$\Pi^s(x) := \begin{cases} \dfrac{x(s)}{1 - \sum_{i \in M \setminus M_s} x_i} & \text{if } \sum_{i \in M \setminus M_s} x_i \neq 1 \\ 0 & \text{if } \sum_{i \in M \setminus M_s} x_i = 1 \end{cases}$$

Note that if x is a strategy in Δ_m, then $\Pi^s(x) = x(s)(\sum_{i \in M_s} x_i)^{-1} \in \Delta_{m_s}$ if $x(s) \neq 0$ and $\Pi^s(x) = 0$ if $x(s) = 0$. Similarly for $y \in \mathbb{R}^n$ and $s \in \{1, \cdots, t\}$ the 'balanced projection of y on $\mathbb{R}^{n_s}$', $\Pi^s(y)$, is defined.
Now we can state

THEOREM 3.1. Let (p, q) be an equilibrium for the major game. Let $s \in \{1, \cdots, t\}$ be such that $p(s) \neq 0$. Then $(\Pi^s(p), \Pi^s(q))$ is an equilibrium for the subgame (A^s, B^s).

Proof. Since $p(s) \neq 0$, there is an $i \in M_s$ such that $e_i A q = \max_{k \in \{1,\cdots,m\}} e_k A q$. In particular this implies $e_{i(s)} A^s q(s) = \max_{k \in M_s} e_{k(s)} A^s q(s)$. Using the definitions of the balanced projections, we then find $C(A^s, \Pi^s(p)) \subset PB(B^s, \Pi^s(q))$. Since, by lemma 3.1, $p(s) \neq 0$ implies $q(s) \neq 0$ we can use similar arguments to show $C(\Pi^s(q)) \subset PB(\Pi^s(p))$, which completes the proof. $\square$

In view of theorem 3.1 an equilibrium for the major game in a natural way defines at least one equilibrium for at least one of the subgames. In the next theorem we, conversely, describe how equilibria of different subgames can yield an equilibrium of the major game. Therefore we need the following definition.

Let S be a subset of $\{1, \cdots, t\}$ and let, for each $s \in S$, (p^s, q^s) be an equilibrium for (A^s, B^s). Then the 'balanced cartesian product' of these equilibria, $\bigotimes_{s \in S} (p^s, q^s)$ is an element of $\Delta_m \times \Delta_n$, say (p, q), which is defined as follows

$$p(s) := \begin{cases} \frac{p^s}{p^s B^s q^s} \left(\sum_{r \in S} \frac{1}{p^r B^r q^r} \right)^{-1} & \text{if } s \in S \\ 0 & \text{if } s \in \{1, \cdots, t\} \backslash S. \end{cases}$$

and

$$q(s) := \begin{cases} \frac{q^s}{p^s A^s q^s} \left(\sum_{r \in S} \frac{1}{p^r A^r q^r} \right)^{-1} & \text{if } s \in S \\ 0 & \text{if } s \in \{1, \cdots, t\} \backslash S. \end{cases}$$

THEOREM 3.2. Let S be a subset of $\{1, \cdots, t\}$ and let, for each $s \in S$, (p^s, q^s) be an equilibrium for the subgame (A^s, B^s). Then $\bigotimes_{s \in S} (p^s, q^s)$ is an equilibrium for the major game.

Proof. Let $(p, q) := \bigotimes_{s \in S} (p^s, q^s)$. Let $i \in C(p)$. Then, by construction of (p, q), $i \in M_s$ for some $s \in S$ and, since (p^s, q^s) is an equilibrium for (A^s, B^s), we have $e_{i(s)} A^s q^s = \max_{k \in M_s} e_{k(s)} A^s q^s$. By definition of q this implies that $e_i A q = \max_{k \in M_s} e_k A q$.

Suppose there is an $i' \in M$ such that $e_{i'} A q > e_i A q$. Then there is an $s' \in S \backslash \{s\}$ such that $i' \in M_{s'}$. Using the definition of q again, we find $\frac{e_{i'(s')} A^{s'} q^{s'}}{p^{s'} A^{s'} q^{s'}} > \frac{e_{i(s)} A^s q^s}{p^s A^s q^s} = 1$. This contradicts $(p^{s'}, q^{s'})$ being an equilibrium for $(A^{s'}, B^{s'})$.

So $e_{i'} A q \leq e_i A q$ for all $i' \in M$. Hence $i \in PB(A, q)$. So we have proved $C(p) \subset PB(A, q)$. Similarly one shows $C(q) \subset PB(B, p)$. $\square$

In view of theorems 3.1 and 3.2 the correspondence between equilibria of the major game and the subgames is complete if we show that every equilibrium of the major game can be created from equilibria of the subgames by means of the balanced cartesian product.

We show this in

THEOREM 3.3. Let (p, q) be an equilibrium of the major game. Then there is a subset S of $\{1, \cdots, t\}$ and, for each $s \in S$, there is an equilibrium (p^s, q^s) of the subgame (A^s, B^s) such that $(p, q) = \bigotimes_{s \in S} (p^s, q^s)$.

Proof. Define $S := \{s \in \{1,\cdots,t\} \mid p(s) \neq 0\}$. Then by lemma 3.1 $p(s) = q(s) = 0$ iff $s \in \{1,\cdots,t\}\backslash S$. Take $s \in S$ and $i \in M_s$ such that $p_i > 0$. Then, since (p,q) is an equilibrium for (A,B),

$$e_{i(s)}A^s q(s) = e_i A q = pAq = \sum_{r\in\{1,\cdots,t\}} p(r)A^r q(r) = \sum_{r\in S} p(r)A^r q(r).$$

Hence

$$p(s)A^s q(s) = \left(\sum_{i\in M_s} p_i\right)\left(\sum_{r\in S} p(r)A^r q(r)\right)$$

For each $r \in \{1,\cdots,t\}$, let $p^r = \Pi^r(p)$ and $q^r = \Pi^r(q)$.
Then the previous equation yields

$$p^s A^s q^s = \left(\sum_{r\in S} p(r)A^r q(r)\right)\left(\sum_{j\in N_s} q_j\right)^{-1},$$

or equivalently

$$\sum_{j\in N_s} q_j = \left(\sum_{r\in S} p(r)A^r q(r)\right)(p^s A^s q^s)^{-1}.$$

Since $\sum\limits_{s\in\{1,\cdots,t\}} \sum\limits_{j\in N_s} q_j = \sum\limits_{s\in S}\sum\limits_{j\in N_s} q_j = 1$ the last equality yields

$$1 = \left(\sum_{r\in S} p(r)A^r q(r)\right)\left(\sum_{s\in S}\frac{1}{p^s A^s q^s}\right).$$

Then we obtain from the last two equalities

$$q(s) = q^s\left(\sum_{j\in N_s} q_j\right) = \frac{q^s}{p^s A^s q^s}\left(\sum_{r\in S}\frac{1}{p^r A^r q^r}\right)^{-1}.$$

Similarly one shows $p(s) = \frac{p^s}{p^s B^s q^s}(\sum\limits_{r\in S}\frac{1}{p^r B^r q^r})^{-1}$.
Since $s \in S$ was arbitrary, we have $(p,q) = \bigotimes\limits_{s\in S}(p^s,q^s)$. $\square$

Finally in this section we return to the example of the introduction. All equilibria of the restaurant game are (m_1,m_2), (m_2,m_1) and $(\frac{1}{2}m_1+\frac{1}{2}m_2, \frac{1}{2}m_1+\frac{1}{2}m_2)$.
The only equilibrium of the 1×1 subgame is (h,h).
Hence all equilibria of the major game are
$(h,h)\otimes(m_1,m_2) = (\frac{2}{3}h+\frac{1}{3}rm_1, \frac{1}{2}h+\frac{1}{2}rm_2)$,
$(h,h)\otimes(m_2,m_1) = (\frac{2}{3}h+\frac{1}{3}rm_2, \frac{1}{2}h+\frac{1}{2}rm_1)$ and
$(h,h)\otimes(\frac{1}{2}m_1+\frac{1}{2}m_2, \frac{1}{2}m_1+\frac{1}{2}m_2) = (\frac{3}{5}h+\frac{1}{5}rm_1+\frac{1}{5}rm_2, \frac{3}{7}h+\frac{2}{7}rm_1+\frac{2}{7}rm_2)$.

4. PERFECT EQUILIBRIA AND STABLE SETS FOR A GAME OF COORDINATION

In this section we consider again the game of coordination (A, B) as defined in the previous section. We show results similar to theorems 3.1 and 3.2 for perfect equilibria and stable sets. First we prove

THEOREM 4.1. Let (p, q) be a perfect equilibrium for the major game. Let $s \in \{1, \cdots, t\}$ be such that $p(s) \neq 0$. Then $(\Pi^s(p), \Pi^s(q))$ is a perfect equilibrium for the subgame (A^s, B^s).

Proof. By theorem 3.1, $(\Pi^s(p), \Pi^s(q))$ is an equilibrium for (A^s, B^s). Since (p, q) is perfect there are sequences $\{(\varepsilon^k, \delta^k)\}_{k\in\mathbb{N}}$ of mistake vectors converging to zero and $\{(p^k, q^k)\}_{k\in\mathbb{N}}$ converging to (p, q) such that $(p^k, q^k) \in E(A(\delta^k), B(\varepsilon^k))$ for every $k \in \mathbb{N}$. Clearly $\{(\Pi^s(\delta^k), \Pi^s(\varepsilon^k))\}_{k\in\mathbb{N}}$ is a sequence of mistake vectors in $\mathbb{R}^{m_s} \times \mathbb{R}^{n_s}$ and $\{(\Pi^s(p^k), \Pi^s(q^k))\}_{k\in\mathbb{N}}$ converges to $(\Pi^s(p), \Pi^s(q))$. We are finished if we can show that $(\Pi^s(p^k), \Pi^s(q^k)) \in E\Big(A^s(\Pi^s(\delta^k)), B^s(\Pi^s(\varepsilon^k))\Big)$ for every $k \in \mathbb{N}$. We only show $C(\Pi^s(p^k)) \subset PB\Big(A^s(\Pi^s(\delta^k)), \Pi^s(q^k)\Big)$. First for $i \in M_s$ we evaluate the expression $e_i A(\delta^k) q^k$ using the special form of A.

$$\begin{aligned}
e_i A(\delta^k) q^k &= \left(1 - \sum_{j\in N} \delta_j^k\right) e_i A q^k + e_i A \delta^k \\
&= \left(1 - \sum_{j\in N} \delta_j^k\right) e_{i(s)} A^s q^k(s) + e_{i(s)} A^s \delta^k(s) \\
&= \left(1 - \sum_{j\in N\setminus N_s} \delta_j^k\right) \left(\frac{1 - \sum_{j\in N} \delta_j^k}{1 - \sum_{j\in N\setminus N_s} \delta_j^k} e_{i(s)} A_q^s k(s) + e_{i(s)} A^s \Pi^s(\delta^k) \right).
\end{aligned}$$

Now $\sum_{j\in\{1,\cdots,n_s\}} \Pi^s(\delta^k)_j = (\sum_{j\in\{1,\cdots,n_s\}} \delta^k(s))(1 - \sum_{l\in N\setminus N_s} \delta_l^k)^{-1} = (\sum_{j\in N_s} \delta_j^k)(1 - \sum_{l\in N\setminus N_s} \delta_l^k)^{-1}$.
Hence

$$\begin{aligned}
e_i A(\delta^k) q^k &= (1 - \sum_{j\in N\setminus N_s} \delta_j^k) \left(\Big(1 - \sum_{j\in\{1,\cdots,n_s\}} \Pi^s(\delta^k)\Big) e_{i(s)} A^s q^k_{(s)} + e_{i(s)} A^s \Pi^s(\delta^k) \right) \\
&= (1 - \sum_{j\in N\setminus N_s} \delta_j^k) e_{i(s)} A^s \big(\Pi^s(\delta^k)\big) q^k(s) \qquad (1)
\end{aligned}$$

By definition of $\Pi^s(q^k)$ it follows from (1) that

$$e_i A(\delta^k) q^k = \left(1 - \sum_{j\in N\setminus N_s} \delta_j^k\right) \left(\sum_{j\in N_s} q_j^k\right) e_{i(s)} A^s \big(\Pi^s(\delta^k)\big) \Pi^s(q^k)$$

Then, for $i \in M_s$ such that $i(s) \in C(\Pi^s(p^k))$ we have $i \in C(p^k)$ and hence $e_i A(\delta^k) q^k = \max_{l\in M} e_l A(\delta^k) q^k = \max_{l\in M_s} e_l A(\delta^k) q^k$. This yields $e_{i(s)} A^s \big(\Pi^s(\delta^k)\big) \Pi^s(q^k) = \max_{l\in M_s} e_{l(s)} A^s \big(\Pi^s(\delta^k)\big) \Pi^s(q^k)$. Hence $C(\Pi^s(p^k)) \subset PB\Big(A(\Pi^s(\delta^k)), \Pi^s(q^k)\Big)$. $\square$

For a result on perfect equilibria similar to theorem 3.2 we need a technical lemma.

LEMMA 4.1. For each $s \in \{1, \cdots, t\}$ let $(\varepsilon_s, \delta_s)$ be a mistake vector in $\mathbb{R}^{m_s} \times \mathbb{R}^{n_s}$. If, for each s, $(\varepsilon_s, \delta_s)$ is close enough to zero, then we can find a mistake vector (ε, δ) in $\mathbb{R}^m \times \mathbb{R}^n$ such that $\Pi^s(\varepsilon) = \varepsilon_s$ and $\Pi^s(\delta) = \delta_s$ for each s.

Proof. We only show that we can find $\delta \in \mathbb{R}^n$ such that $\Pi^s(\delta) = \delta_s$ for each s, while $\sum_{j \in N} \delta_j < 1$ and $\delta > 0$.

First we show that we can find $\lambda := (\lambda_1, \cdots, \lambda_t) \in \mathbb{R}^t$ such that for each $s \in \{1, \cdots, t\}$ we have $\delta(s) = \lambda_s \delta_s$. By definition of $\Pi^s(\delta)$ we must have $\lambda_s = 1 - \sum_{j \in N \setminus N_s} \delta_j$ for each s. Define, for $s \in \{1, \cdots, t\}$, $\Delta_s := \sum_{j \in \{1, \cdots, n_s\}} \delta_{s_j}$. Then $\sum_{j \in N_s} \delta_j = \lambda_s \Delta_s$ for $s \in \{1, \cdots, t\}$ and hence $1 - \lambda_s = \sum_{j \in N \setminus N_s} \delta_j = \sum_{r \neq s} \lambda_r \Delta_r$, or equivalently $1 = \sum_{r \neq s} \lambda_r \Delta_r + \lambda_s$.

In matrix notation this is $(I + \Delta)\lambda = 1_t$, where I is the $t \times t$ identity matrix, $1_t \in \mathbb{R}^t$ has all entries equal to 1 and Δ is the $t \times t$ matrix

$$\Delta := \begin{bmatrix} 0 & \Delta_2 & \dots & \Delta_{t-1} & \Delta_t \\ \Delta_1 & 0 & & & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & 0 & \Delta_t \\ \Delta_1 & \Delta_2 & \dots & \Delta_{t-1} & 0 \end{bmatrix}$$

Now if, for each $s \in S$, δ_s is close to zero in $\mathbb{R}^{n_s}$, then Δ_s is close to zero and $(I + \Delta)$ is invertible.

In fact for small entries of Δ we have $(I + \Delta)^{-1} = \sum_{l=0}^{\infty} (-\Delta)^l$, so that $\lambda = 1_t + \sum_{l=1}^{\infty} (-\Delta)^l 1_t$. Hence, in that case, for each $s \in \{1, \dots, t\}$, we have $0 < \lambda_s < 1$, so that $\delta(s) = \lambda_s \delta_s > 0$. Moreover small entries of Δ also imply that $\sum_{j \in N} \delta_j = \sum_r \lambda_r \Delta_r < 1$. □

THEOREM 4.2. Let S be a subset of $\{1, \dots, t\}$ and let, for each $s \in S$, (p^s, q^s) be a perfect equilibrium for the subgame (A^s, B^s). Then $\bigotimes_{s \in S} (p^s, q^s)$ is a perfect equilibrium for the major game.

Proof. Since (p^s, q^s) is an equilibrium for (A^s, B^s) for each $s \in S$, theorem 3.2 implies that $(p, q) := \bigotimes_{s \in S} (p^s, q^s)$ is an equilibrium for (A, B). Since (p^s, q^s) is perfect for every $s \in S$, it follows that there are, for every $s \in S$, sequences $\{(\varepsilon_s^k, \delta_s^k)\}_{k \in \mathbb{N}}$ of mistake vectors in $\mathbb{R}^{m_s} \times \mathbb{R}^{n_s}$ converging to zero and $\{(p^{sk}, q^{sk})\}_{k \in \mathbb{N}}$ converging to (p^s, q^s) such that $(p^{sk}, q^{sk}) \in E(A^s(\delta_s^k), B^s(\varepsilon_s^k))$ for every k.

For $r \in \{1, \dots, t\} \setminus S$ let $\{(\varepsilon_r^k, \delta_r^k)\}_{k \in \mathbb{N}}$ be an arbitrary sequence of mistake vectors in $\mathbb{R}^{m_r} \times \mathbb{R}^{n_r}$ converging to zero. Then, by lemma 4.1, there is a sequence of mistake vectors $\{(\varepsilon^k, \delta^k)\}_{k \in \mathbb{N}}$ in $\mathbb{R}^m \times \mathbb{R}^n$ such that, for large k, $\Pi^r(\varepsilon^k) = \varepsilon_r^k$ and $\Pi^r(\delta^k) = \delta_r^k$ for every $r \in \{1, \dots, t\}$. Consequently this sequence converges to zero. Now define $q^k \in \Delta_n$ by

$$q^k(s) = \begin{cases} \frac{q^{sk}(1 - \sum_{j \in N \setminus N_s} \delta_j^k)^{-1}}{p^{sk} A^s(\Pi^s(\delta^k)) q^{sk}} \left(\sum_r \frac{(1 - \sum_{j \in N \setminus N_r} \delta_j^k)^{-1}}{p^{rk} A^r(\Pi^r(\delta^k)) q^{rk}} \right)^{-1} & \text{if } s \in S \\ 0 & \text{if } s \notin S \end{cases}$$

Similarly define $p^k \in \Delta_m$. Then the sequence $\{(p^k, q^k)\}_{k\in\mathbb{N}}$ converges to (p, q). So if we show that $(p^k, q^k) \in E(A(\delta^k), B(\varepsilon^k))$ for every $k \in \mathbb{N}$, then we are finished. We only show $C(p^k) \subset PB(A(\delta^k), q^k)$. Let $k \in \mathbb{N}$.
Similar to the evaluation (1) we find for $i \in M_s$ and $s \in S$

$$e_i A(\delta^k) q^k = \left(1 - \sum_{j \in N\setminus N_s} \delta_j^k\right) e_{i(s)} A^s(\delta_s^k) q^k(s).$$

Filling in the expression for $q^k(s)$, we obtain

$$e_i A(\delta^k) q^k = e_{i(s)} A^s(\delta_s^k) \frac{q^{sk}}{p^{sk} A^s(\delta_s^k) q^{sk}} \left(\sum_r \frac{1 - \sum_{j\in N\setminus N_r} \delta_j^k}{p^{rk} A^r(\delta_r^k) q^{rk}} \right)^{-1}.$$

Suppose $i \in C(p^k)$. Then clearly $i(s) \in C(p^{sk})$ for some $s \in S$ and since $C(p^{sk}) \subset PB(A(\delta_s^k), q^{sk})$ we find that, if for some $l \in \{1, \dots, m\}$ we have $e_l A(\delta^k) q^k > e_i A(\delta^k) q^k$, then $l \notin M_s$. So $l \in M_{s'}$ for some $s' \in S\setminus\{s\}$. Then

$$\frac{e_{l(s')} A^{s'}(\delta_{s'}^k) q^{s'k}}{p^{s'k} A^s(\delta_{s'}^k) q^{s'k}} > \frac{e_{i(s)} A^s(\delta_s^k) q^{sk}}{p^{sk} A^s(\delta_s^k) q^{sk}} = 1,$$

which is a contradiction since $(p^{s'k}, q^{s'k}) \in E(A^{s'}(\delta_{s'}^k), B^{s'}(\varepsilon_{s'}^k))$. So $e_l A(\delta^k) q^k \le e_i A(\delta^k) q^k$ for all $l \in M$, so that $C(p^k) \subset PB(A(\delta^k), q^k)$. $\square$

Next we discuss stable sets, but first we need some definitions. Let T be a subset of the set of equilibria for the major game. Then the 'balanced projection of T on $\Delta_{m_s} \times \Delta_{n_s}$' for $s \in \{1, \dots, t\}$ is defined as

$$\Pi^s(T) := \{(\Pi^s(p), \Pi^s(q)) \mid (p, q) \in T\}.$$

Let S be a subset of $\{1, \dots, t\}$ and let, for each $s \in S$, T^s be a subset of the set of equilibria for the subgame (A^s, B^s). Then the 'balanced cartesian product' of these sets is defined as

$$\bigotimes_{s\in S} T^s := \left\{ \bigotimes_{s\in S} (p^s, q^s) \mid (p^s, q^s) \in T^s \text{ for each } s \in S \right\}.$$

Then we have

LEMMA 4.2 (i) Let T be a strictly perfect set for the major game. Then for each $s \in \{1, \dots, t\}$ such that $\Pi^s(T) \neq 0$, $\Pi^s(T)$ is a strictly perfect set for the subgame (A^s, B^s).
(ii) Let S be a subset of $\{1, \dots, t\}$ and let, for each $s \in S$, T^s be a strictly perfect set for the major game.

Proof. (i) Let $s \in \{1, \dots, t\}$ be such that $\Pi^s(T) \neq 0$. Let, for $r \in \{1, \dots, t\}$, $\{(\varepsilon_r^k, \delta_r^k)\}_{k\in\mathbb{N}}$ be a sequence of mistake vectors in $\mathbb{R}^{m_r} \times \mathbb{R}^{n_r}$ converging to zero. In view of lemma 4.1 there is a sequence $\{(\varepsilon^k, \delta^k)\}_{k\in\mathbb{N}}$ of mistake vectors in $\mathbb{R}^m \times \mathbb{R}^n$ such that $\Pi^r(\varepsilon^k) = \varepsilon_r^k$ and $\Pi^r(\delta^k) = \delta_r^k$ for large k. Then this sequence converges to zero. So in view of the strictly perfectness of T there is a sequence $\{(p^k, q^k)\}_{k\in\mathbb{N}}$ with a limit point in T such that $(p^k, q^k) \in E(A(\delta^k), B(\varepsilon^k))$ for every k. Then $\{(\Pi^s(p^k), \Pi^s(q^k))\}_{k\in\mathbb{N}}$ has a limit point in T^s. Then as in the proof of theorem 4.1 one can show that $(\Pi^s(p^k), \Pi^s(q^k)) \in E(A^s(\delta_s^k), B^s(\varepsilon_j^k))$ for large k. Since $\{(\varepsilon_s^k, \delta_s^k)\}_{k\in\mathbb{N}}$ is chosen arbitrary, this shows that T^s is strictly perfect.

(ii) Let $\{(\varepsilon^k, \delta^k)\}_{k\in\mathbb{N}}$ be a sequence of mistake vectors in $\mathbb{R}^m \times \mathbb{R}^n$ converging to zero. Then, for $s \in S$, $\{(\Pi^s(\varepsilon^k), \Pi^s(\delta^k)\}_{k\in\mathbb{N}}$ is a sequence of mistake vectors in $\mathbb{R}^{m_s} \times \mathbb{R}^{n_s}$ converging to zero. The strictly perfectness of T^s for each $s \in S$ implies the existence, for each $s \in S$, of a sequence $\{(p^{sk}, q^{sk})\}_{k\in\mathbb{N}}$ with a limit point in T^s such that $(p^{sk}, q^{sk}) \in E(A^s(\Pi^s(\delta^k)), B^s(\Pi^s(\varepsilon^k)))$ for every $k \in \mathbb{N}$. Defining a sequence $\{(p^k, q^k)\}_{k\in\mathbb{N}}$ as in the proof of theorem 4.2 it will be clear that this sequence has a limit point in $T := \bigotimes_{s\in S} T^s$. Further proceeding as in the proof of theorem 4.2 one shows that $(p^k, q^k) \in E(A(\delta^k), B(\varepsilon^k))$ for every $k \in \mathbb{N}$, which implies that T is strictly perfect. $\square$

Suppose T is a stable set for the major game. Let $S := \{s \in \{1, \ldots, t\} \mid \Pi^s(T) \neq 0\}$. Then, in view of theorem 3.3, $T = \bigotimes_{s\in S} \Pi^s(T)$. By lemma 4.2(i), $\Pi^s(T)$ is strictly perfect for each $s \in S$. Suppose, for $s \in S$, T^s is a stable set contained in $\Pi^s(T)$. Then by lemma 4.2(ii) $\bigotimes_{s\in S} T^s$ is strictly perfect. However, $\bigotimes_{s\in S} T^s \subset \bigotimes_{s\in S} \Pi^s(T) = T$, and since T is stable, we must have $\bigotimes_{s\in S} T^s = \bigotimes_{s\in S} \Pi^s(T)$, or equivalently, $T^s = \Pi^s(T)$ for every $s \in S$. Hence $\Pi^s(T)$ is stable for each $s \in S$. A similar reasoning shows that $\bigotimes_{s\in S} T^s$ is stable if T^s is stable for each $s \in S$. Hence

THEOREM 4.3. Lemma 4.2(i) and (ii) also hold if everywhere 'strictly perfect' is replaced by 'stable'.

Returning again to the example of section 1, we note that for the restaurant game all three equilibria are perfect and even strictly perfect. This implies, in view of theorem 4.3, that all equilibria of the major game are strictly perfect, since the 'home game' has only one equilibrium.

5. FINAL REMARKS

In Jansen (1981) it is shown that the set of equilibria for a bimatrix game is the union of a finite number of maximal Nash subsets. One can show that between these sets for the major game and for the subgames there is a correspondence similar to the one in theorems 3.1-3.3. With respect to other refinements we can remark that theorems 4.1-4.2 also hold if 'perfect' is replaced by quasi-strong (cf. Harsanyi (1973)).

Consider a game of coordination as in section 3 for which $t = 2$. Let us exchange the sets N_1 and N_2 for the major game. Then we arrive at

$$(A, B) = \begin{bmatrix} (0,0) & (A^1, B^1) \\ (A^2, B^2) & (0,0) \end{bmatrix}.$$

Assume A^2 is the transpose of B^1 and B^2 of A^1. Then (A, B) is the symmetric game occuring in Griesmer, Hoffman and Robinson (1963) and Jansen, Potters and Tijs (1986), which is a 'symmetrization' of the game (A^1, B^1). For this case the results in section 3 and theorems 4.1 and 4.2 were already obtained by Jansen, Potters and Tijs.

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